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Spatial oligopolies with cooperative distribution

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ABSTRACT. The main objects here are Nash equilibria in spatial Cournot oligopolies when profits depend on coordinated distribution. Production is non-cooperative, but the subsequent transportation must be performed jointly to minimize costs. Cournot-Nash equilibria for this two-stage game with partial coalitional strategies are determined by means of a mathematical-based algorithm. A numerical illustration is presented.

Key words: spatial oligopolies; core allocation; Nash equilibrium; algorithms

JEL classification: C61, C71, C72, L13

1. INTRODUCTION

This paper considers an oligopolistic industry comprising geographically separated firms and markets. Those firms interact at two stages. First, they produce, possibly at many locations, commodities to be delivered at several shared markets. Second, if possible, they agree on how to best distribute the goods to satisfy consumers' demand. More specifically, ex ante firms decide, independently and without collaboration, how much of various commodities to produce and supply at different markets. When making these decisions, they know that all goods will finally be transported from production sites to customers in an efficient manner.

An increasing number of deregulated network industries – including electricity and natural gas supply – are examples where competition may occur in this way: Firms often have market power in production and sales – they produce non-cooperatively – whereas product distribution is regulated to enhance efficiency or mitigate monopoly effects, see e.g., Newbery (1999) for an overview of competitive issues regarding network industries. For practical purposes the (regulated) coordinated activities may either be subject to contracts or outsourced to competitive agents, offering appropriately specialized services.

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The obvious questions is then: Will the overall two-stage game admit an equilibrium? Affirmative and constructive answers are already given in a recent study by Flåm and Jourani (2003), who define and characterize equilibria in games of this kind. Methods for determining an equilibrium solution in practice are, however, lacking. The purpose of the present paper is, therefore, to present a new characterization of equilibrium of the spatial oligopoly game featuring both non-cooperative and cooperative strategies and, moreover, to provide algorithms for determining an equilibrium solution. In doing so, I extend the mathematical programming approach of Murphy, Sherali and Soyster (1982) to a spatial and partial coalitional context.

The two-stage game outlined above partly belongs to regional science. The study of spatial competition was initiated by Samuelson (1952). Harker (1986) reviews the early literature, formulates equilibria by variational inequalities, and brings out algorithms that solve the latter; see also Nagurney (1988) and Miller, Tobin and Friesz (1991). In all these studies, however, competition is purely non-cooperative, while insisting on joint distribution, as in the present paper, requires adoption of methods from cooperative game theory.

Whether coordinated transportation is voluntary or imposed by the authorities, it had better reflect effort to minimize cost. The joint transportation problem will, therefore, be formalized here as a cooperative game in which the cost allocation constitutes a core solution (Gillies 1953). Each firm reasonably anticipates, during the production phase, that the said allocation will indeed belong to the core. More specifically, the second-stage coalitional game fits the form of so-called production games (Owen 1975), which belong to the large family of flow games; see Kalai and Zemel (1982a&b) and Dubey and Shapley (1984). Moreover, Owen (1975) constructively finds a core allocation using any optimal solution of the dual linear program associated to the grand coalition. (For generalizations and extensions of production games see e.g. Granot 1986, Sandsmark 1999 and Evstigneev and Flåm 2001). His recipe perfectly fits our setting.

The paper is organized as follows. Section 2 presents the two-stage model, in line with the regional oligopoly model of Flåm and Jourani (2003). Reviewed in that section are the basic concepts from cooperative game theory. In Section 3 the overall Cournot-Nash equilibrium is established and characterized. Two related algorithms for determining the equilibrium solution are presented in Section 4, along with a numerical illustration. Section 5 concludes.

2. The two-stage model

Throughout there is a fixed, finite set I of firms with production sites located at one or more origins $o \in O$. (It is common, e.g., for electricity companies − both small and large − to own power generating units at several nodes of a transmission network.) A firm does not, however, need to have a factory at each origin o. All firms produce a homogenous good sold to customers distributed at various destinations $d \in D$. By assumption the sets O, D are finite and disjoint. The chief economic activities of the firms take place over two stages: first comes production, second is transportation.

These are discussed next in that order.

2.1. First-stage non-cooperative production. Firm $i \in I$ first decides on a production plan $q_i^{od} \in \mathbb{R}_+^{O \cup D}$, where $q_{io} := \sum_{d \in D} q_i^{od}$ is the amount to be produced at origin o and $q_{id} := \sum_{o \in O} q_i^{od}$ is the amount to be sold to consumers at destination d, subject, of course, to the balance condition

$$\sum_{d \in D} q_{id} = \sum_{o \in O} q_{io}.$$

Let $Q_d := \sum_{i \in I} q_{id}$ denote the total supply at destination d. Demand there is embodied in a non-increasing, differentiable inverse curve $P_d(Q_d)$ that states the unit price at which local consumers will purchase the amount Q_d. I posit that gross revenue $q_{id} \mapsto P_d(Q_d)Q_d$ be concave for each d.

Further, production cost $f_{io}(q_{io})$ of firm i at origin o is taken to be convex. It is tacitly understood that $f_{io}(q_{io}) = +\infty$ if $q_{io} < 0$ or if q_{io} exceeds some specified capacity. Firm i's payoff $\pi_i(q) \in \mathbb{R} \cup \{-\infty\}$ from sales is then given by

$$\pi_i(q) = \pi_i(q_i, q_{-i}) := \sum_{d \in D} P_d(Q_d)\, q_{id} - \sum_{o \in O} f_{io}(q_{io}). \tag{1}$$

It depends on own choice $q_i := (q_{io}, q_{id})_{o \in O, d \in D}$ and the output profile q_{-i} which is shorthand for the choices $q_j := (q_{jo}, q_{jd})_{o \in O, d \in D}$, $j \in I \backslash i$, made by i's rivals.

The key objects here are equilibrium outcomes. Before turning to those we must describe

2.2. Second-stage cooperative distribution. For firms to enjoy sale proceeds, the goods produced must be shipped to consumers. Assume all firms incur transportation cost c_{od} per unit shipped from production site o to destination d. The unit cost c_{od}^i might very well vary across firms. For notational simplicity we shall stick to the uniform case though. Fixed cost are ignored. Transportation costs are thus proportional to the amount shipped. Further, I assume that the transportation cost matrix does not allow for the transportation paradox, c.f. Dejneko, Klinz and Woegniger (2003).

Let x_{od} denote the amount of goods shipped from origin o to destination d. Then the second-stage problem of firm i would, in autarky, amount to finding a distribution pattern $x \in \mathbb{R}_+^{O \times D}$ that minimizes its transportation costs. This is done by solving

$$\left.\begin{array}{lllll} \text{minimize} & \sum_{o \in O, d \in D} c_{od} x_{od} & & & \\ \text{subject to} & \sum_{d \in D} x_{od} & \leq & q_{io}, & \text{for all } o, \\ & \sum_{o \in O} x_{od} & \geq & q_{id}, & \text{for all } d, \\ \text{and} & x_{od} & \geq & 0, & \text{for all } o, d. \end{array}\right\} \tag{2}$$

The first inequality of (2) implies that no firm can ship more goods from a production site o than what it makes available there; the second equation tells that the total amount of goods it ships to destination d must meet the obligation (commitment) to serve there. Denote the minimal cost in (2) by $c(q_i)$.

Since the products are homogeneous, customers are indifferent as to the origins of goods. As a result there are potential efficiency gains to be had from aggregating individual transportation tasks. Upon doing so, some firms can supply nearby customers on behalf of more remote firms. For the argument suppose coalition $S \subseteq I$ of firms were to coordinate their transport to customers. After pooling their supply-demand vectors to have aggregates

$$q_{So} := \sum_{i \in S} q_{io} \text{ and } q_{Sd} := \sum_{i \in S} q_{id},$$

they could find the distribution pattern $x \in \mathbb{R}_+^{O \times D}$ that

$$\left.\begin{array}{llll} \text{minimize} & \sum_{o \in O, d \in D} c_{od} x_{od} & & \\ \text{subject to} & \sum_{d \in D} x_{od} & \leq q_{So}, & \text{for all } o, \\ & \sum_{o \in O} x_{od} & \geq q_{Sd}, & \text{for all } d, \\ \text{and} & x_{od} & \geq 0, & \text{for all } o, d. \end{array}\right\} \quad (3)$$

Let $c(q_S)$ denote the minimal cost in (3). Our concern is whether the overall minimal cost $c(q_I)$ can be achieved and fairly divided. For that issue consider the *cooperative transferable utility game* with characteristic function $S \mapsto c(q_S)$. This game (I, c) has *orthogonal coalitions,* meaning that members of S can achieve cost $c(q_S)$ regardless of what players outside S do. That is, the only threat against S by a player $i \in I \backslash S$, or any coalition of such players, is the boycott (see e.g. Shubik, 1982). Further, a reasonable allocation $u = (u_i)_{i \in I}$ of total costs $c(q_I)$ should lie in the *core.*

Definition 1. *(Core) A cost allocation $u = (u_i)_{i \in I}$ is an element in the core of the cooperative game (I, c) if*

$$\sum_{i \in S} u_i \leq c(q_S), \text{ for all } S \subset I, \quad \text{and} \quad \sum_{i \in I} u_i = c(q_I).$$

The inequalities imply coalitional stability: no single or group of players can do better by themselves. The equation accounts for Pareto efficiency. Subadditivity is necessary for non-emptyness of the core:

$$c(q_S) + c(q_{S'}) \geq c(q_S + q_{S'}) \text{ for disjoint coalitions } S, S' \subset I.$$

This condition evidently holds in our case. Moreover, the cost sharing game is balanced, which suffices for non-emptiness of the core, c.f. Bondareva (1962) and Shapley (1967). From the analysis of production games in Owen (1975) we get forthwith

Theorem 1. *(Non-empty core of second-stage transportation game) Assume that for any society-wide profile $q = (q_i)_{i \in I}$ decided upon during the first stage, the subsequent linear program (3) is feasible and has a finite optimal value for $S = I$. Then (3) defines a cooperative transportation game which is totally balanced.* □

Most important, Owen (1975) constructively displayed a core allocation in terms of solutions associated with the dual to program (3) when $S = I$. Of course, to make that result applicable here, we must require that the optimal value $c(q_I)$ be finite for all strategy profiles $q = (q_i)_{i\in I}$ that might come into consideration. Then core elements are easily found. To wit, let $\lambda := (\lambda_d, \lambda_o)_{d\in D, o\in O}$ be any optimal solution to the dual transportation problem:

$$\left.\begin{array}{lll} \text{maximize} & \sum_{d\in D}\lambda_d q_{Id} - \sum_{o\in O}\lambda_o q_{Io} & \\ \text{subject to} & \lambda_d - \lambda_o & \leq \; c_{od} \;\; \text{for all} \;\; o,d, \quad \lambda \geq 0. \end{array}\right\}$$

Then we have

$$c(q_I) \; = \; \sum_{d\in D}\lambda_d q_{Id} - \sum_{o\in O}\lambda_o q_{Io} \text{ and} \tag{4}$$

$$c(q_S) \; \geq \; \sum_{d\in D}\lambda_d q_{Sd} - \sum_{o\in O}\lambda_o q_{So}, \text{ for any } S. \tag{5}$$

Consequently, distributing total costs $c(q_I)$ by the rule

$$u_i := \sum_{d\in D}\lambda_d q_{id} - \sum_{o\in O}\lambda_o q_{io}, \text{ for all } i \in I,$$

we have, for any S,

$$\sum_{i\in S} u_i = \sum_{i\in S}\left(\sum_{d\in D}\lambda_d q_{id} - \sum_{o\in O}\lambda_o q_{io}\right)$$

yielding by (4) and (5), respectively,

$$\sum_{i\in I} u_i = c(q_I) \text{ and } \sum_{i\in S} u_i \leq c(q_S)$$

Thus the resulting allocation $u = (u_i)_{i\in I}$ belongs to the core of (3). The relationship between the core of games generated from linear programming problems and the set of dual optimal solutions, is further studied in Samet and Zemel (1984). An axiomatic characterization of the set of solutions derived in this manner is presented in Van Gellenkom et al. (2000).

To sum up, pooling supply-demand vectors $q_i = (q_{io}, q_{id})_{o\in O, d\in D}$ and solving (3) for $S = I$, yields an optimal distribution pattern x^*. By implementing this solution, firms incur joint minimum transportation cost $c(q_I)$, and firm i's second-stage share of the joint transportation cost is defined as

$$u_i(q) := \sum_{d\in D}\lambda_d q_{id} - \sum_{o\in O}\lambda_o q_{io} \tag{6}$$

When costs are shared as suggested above, each firm pays in accordance with the transportation task it brings to the community evaluated by the optimal Lagrangian multipliers. For convenience and in anticipation of the subsequent analysis, I introduce the set

$$\Lambda(q) := \{\text{all } \lambda \text{ which are dually optimal in (3) for } S = I\}.$$

Proposition 1. *The function* $q \to u(q) := \sum_{i\in I} u_i(q_i) = c(q_I)$ *is convex and* $\partial u(q)/\partial q_i = \partial u_i(q_i)/\partial q_i = \Lambda(q)$. *Moreover, the correspondence* $q \to \Lambda(q)$ *is monotone.*

Proof. The function

$$(x,q) \in \mathbb{R}^{O\times D} \times \mathbb{R}^{O\cup D} \to C(x,q) := \begin{cases} \sum_{o,d} c_{od} x_{od} \text{ if } \begin{cases} \sum_d x_{od} & \le & q_o, \\ \sum_o x_{od} & \ge & q_d, \\ x_{od} & \ge & 0, \forall o,d \end{cases} \\ +\infty \text{ otherwise} \end{cases}$$

is convex, hence so is $c(q) := \inf_x C(x,q)$. By duality we have

$$c(q) = \sup\left\{ \sum_d \lambda_d q_d - \sum_o \lambda_o q_o : \lambda_o, \lambda_d \in \mathbb{R}_+ , \lambda_d - \lambda_o \le c_{od}, \forall o, d \right\}$$

and therefore,

$$\partial c(q) = \Lambda(q),$$

where $\partial c(q)$ denotes the generalized subdifferential of convex analysis (Rockafellar, 1970). Any such subdifferential is monotone. Since

$$c(q_I) := \inf\left\{ \sum_{i\in I} c(q_i) : \sum_{i\in I} q_i = q_I \right\}$$

is a so-called *inf-convolution*, it follows that $\partial c(q_I) = \partial c(q_i)$ for all i. □

3. Equilibrium for the two-stage game

Assembling the two stages, let

$$\Pi_i(q) := \pi_i(q) - u_i(q), \tag{7}$$

denote firm i's overall profit. It seeks to maximize $\Pi_i(q)$ with respect to own choice q_i while anticipating q_{-i}. The first-stage profit $\pi_i(q)$, that is revenues from sales minus production costs, was defined in (1), and the individual share of second-stage joint transportation cost $u_i(q)$ was defined in (6). We can now establish an equilibrium for the spatial two-stage oligopoly game.

Definition 2. *(Equilibrium) A strategy profile* $\bar{q} = (\bar{q}_i)_{i\in I} \ge 0$ *constitutes a Cournot-Nash equilibrium with partial coalitional strategies if each* $\bar{q}_i$ *is an optimal solution to the problem*

$$\underset{q_i \ge 0}{\text{maximize}}\ \Pi_i(\cdot, \bar{q}_{-i}), \text{ for all } i \in I. \tag{8}$$

Recall that by assumption the individual objective (7) is concave in q_i and jointly continuous in the profile q. Therefore existence of equilibrium is not difficult to ensure:

Theorem 2. *(Existence of equilibrium, Flåm and Jourani 2003) Suppose that each q_i belongs to a nonempty compact convex set $\mathbb{Q}_i \subset \mathbb{R}_+^{O\times D}$. Then there exists at least one Nash equilibrium.* □

Consider now the following characterization of equilibrium.

Theorem 3. *(Characterization of equilibrium) Assume the industry revenue curve $P_d(Q_d)Q_d$ is concave for $Q_d \geq 0$. Then a strategy profile $\bar{q} = (\bar{q}_i)_{i\in I} \geq 0$ is a Cournot-Nash equilibrium if and only if there exists $\lambda \in \Lambda(\bar{q})$ such that*

$$\begin{array}{ll} P_d(\bar{Q}_d) + P_d'(\bar{Q}_d)\bar{q}_{id} - \lambda_d - f_{io}'(\bar{q}_{io}) + \lambda_o \leq 0 & \text{for all } o, d, \text{ for each } i \\ \bar{q}_i\left[P_d(\bar{Q}_d) + P_d'(\bar{Q}_d)\bar{q}_{id} - \lambda_d - f_{io}'(\bar{q}_{io}) + \lambda_o\right] = 0 & \text{for all } o, d, \text{ for each } i, \text{ where} \\ \sum_i \bar{q}_{id} = \bar{Q}_d, \quad \text{for all } d & \\ \bar{q}_i \geq 0 \quad \text{for all } i. & \end{array}$$

Proof. From Theorem 1 we know that there exists a feasible vector of dual optimal values $\lambda = (\lambda_o, \lambda_d)_{o\in O, d\in D}$ associated with individual share of joint transportation costs (6) for any choice $q = (q_i)_{i\in I}$. Further, given Proposition 1 above and Lemma 1 of Murphy, Sherali and Soyster (1982) the objective function of (8) is concave. Moreover, Theorem 3 states the Kuhn-Tucker conditions for this problem, which then are both necessary and sufficient for optimality of individual choice. □

Now, denote by $MR_{id}(q)$ firm i's marginal revenue at destination d and by $MC_{io}(q)$ firm i's marginal cost at origin c, such that

$$MR_{id}(q) := P_d(Q_d) + P_d'(Q_d)q_{id} - \lambda_d \text{ and } MC_{io}(q) := f_{io}'(q_{io}) + \lambda_o$$

Then, from the conditions stated in Theorem 3, it follows that in equilibrium each firm i which has all $\bar{q}_i > 0$, has marginal revenue at each d equal to marginal cost at each o.

Corollary 1. *(Characterization of equilibrium) When $\bar{q} = (\bar{q}_i)_{i\in I} > 0$ is a Cournot-Nash equilibrium with partial coalitional strategies it holds that*

$$MR_{id}(\bar{q}) = MC_{io}(\bar{q}) \quad \text{for all } o, d \text{ for each } i \in I. \qquad \square$$

Using the results derived above, the next section provides two related methods for determining an equilibrium solution to the spatial oligopoly game with cooperative distribution.

3.1. Determining an equilibrium solution. In order to find individual strategies that satisfy Theorem 3, regard all numbers Q_d as constant parameters and consider the problem

$$\left.\begin{array}{ll} \max \sum\limits_{i\in I} \left\{\sum_{d\in D}\left([P_d(Q_d)-\lambda_d]\,q_{id}+\frac{1}{2}P_d'(Q_d)q_{id}^2\right)-\sum_{o\in O}\left(f_{io}\left(q_{io}\right)-\lambda_o q_{io}\right)\right\} \\ \text{subject to} \quad \left\{\begin{array}{lcll} \sum_i q_{id} & = & Q_d & \text{for all } d, \text{ and} \\ q_i & \geq & 0 & \text{for all } i. \end{array}\right. \end{array}\right\} \tag{9}$$

Murphy, Sherali and Soyster (1982) constructed the above equilibrating problem for the non-spatial case, i.e., with production and sales taking place within a singleton, and, also, without joint second-stage transportation activities. The method by which an algorithm is constructed from this problem is, however, equivalent.

The objective of (9) is concave, the constraints are linear, and, due to $P_d'(Q_d)<0$ for all $d\in D$, the maximization is taken over a compact convex set. Hence an optimal solution exists. Then consider the Kuhn-Tucker conditions for this problem, where γ_d and $-\mu_i$ denote the multipliers associated with the constraints of (9) for all d and i, respectively:

$$\left.\begin{array}{lcll} P_d(Q_d)+P_d'(Q_d)q_{id}-\lambda_d-f_{io}'(q_{io})+\lambda_o-\gamma_d+\mu_i=0 & & & \text{for all } o,d, \text{ for each } i \\ \sum_i q_{id} & = & Q_d, & \text{for all } d \\ \mu_i q_i & = & 0 & \text{for all } i, \text{ and} \\ \mu_i, q_i & \geq & 0 & \text{for all } i. \end{array}\right\} \tag{10}$$

These conditions are both necessary and sufficient for optimality. The usefulness of program (9) is now asserted in the following proposition.

Proposition 2. *Let $Q_d^*\geq 0$ be such that the optimal solution $q^*=(q_i^*)_{i\in I}$ to (9) satisfies, for some $\lambda\in\Lambda(q^*)$, the Kuhn-Tucker conditions above with $\gamma_d=0$ for all $d\in D$. Then the profile $q^*=(q_i^*)_{i\in I}$ is a Cournot-Nash equilibrium with partial coalitional strategies. Conversely, let $\bar{q}=(\bar{q}_i)_{i\in I}$ be an equilibrium solution for some $\lambda\in\Lambda(\bar{q})$. Then the profile $\bar{q}=(\bar{q}_i)_{i\in I}$ solves (9) where $\bar{Q}_d=\sum_i\bar{q}_{id}$ for all $d\in D$. Moreover, if $\bar{Q}_d>0$, then γ_d is necessarily zero.*

Proof. If the program (9) yields $\gamma_d=0$ with Q_d^* for all $d\in D$, and some $\lambda\in\Lambda(q^*)$, then we can use (10) to verify that the conditions stated in Theorem 3 are satisfied with $\bar{Q}_d$ replaced by Q_d^* for all $d\in D$ and $\bar{q}=(\bar{q}_i)_{i\in I}$ replaced by $q^*=(q_i^*)_{i\in I}$. Consequently, $q^*=(q_i^*)_{i\in I}$ is an equilibrium solution. From Theorem 3 the solution $q_i=\bar{q}_i$ for all i, $\lambda\in\Lambda(\bar{q})$, $\gamma_d=0$ for all $d\in D$, and $u_i=f_{io}'(\bar{q}_{io})+\lambda_d-P_d(\bar{Q}_d)-P_d'(\bar{Q}_d)\bar{q}_{id}-\lambda_o$ satisfies (10) with $Q_d=\bar{Q}_d$ for all $d\in D$. Therefore, $\bar{q}=(\bar{q}_i)_{i\in I}$ solves (9) with $\bar{Q}_d$ for all $d\in D$. Moreover, for any equilibrium solution $\bar{q}=(\bar{q}_i)_{i\in I}\geq 0$ there exists a firm i for which

$$P_d(\bar{Q}_d)+P_d'(\bar{Q}_d)\bar{q}_{id}-\lambda_o=f_{io}'(\bar{q}_{io})+\lambda_d \quad \text{for all } o \text{ and } d.$$

Since $\bar{q} = (\bar{q}_i)_{i\in I}$ solves (9) with $\bar{Q}_d$ for all $d \in D$, then $u_i = 0$ which, in turn, means that $\gamma_d = 0$ for all $d \in D$. □

Determining an equilibrium profile by this procedure, one has to find an optimal vector $\lambda = (\lambda_d, \lambda_o)_{d\in D, o\in O}$ associated with the numbers Q_d, for all $d \in D$, before checking whether these particular values Q_d yield multipliers γ_d equal to zero or not. Therefore, if any $\gamma_d \neq 0$ the corresponding Q_d must be modified, and, consequently, the optimal λ will also change.

Alternatively, regard all numbers Q_d as constant parameters and recall that shipment along the route $o \to d$ induces unit cost c_{od}. Then consider another auxiliary program

$$\left.\begin{array}{ll} \text{maximize} & \sum\limits_{i\in I} \left\{ \sum_{d\in D} P_d(Q_d) q_{id} + \frac{1}{2} P_d'(Q_d) q_{id}^2 - \sum_{o\in O} f_{io}(q_{io}) \right\} - cx \\ \text{subject to} & \left\{ \begin{array}{l} \sum_i q_{id} = Q_d \ \text{ for all } d, \\ \sum_d x_{od} \le q_{Io} \ \text{ for all } o, \\ \sum_o x_{od} \ge q_{Id} \ \text{ for all } d, \\ \text{and } \ q_i, \ x_{od} \ge 0 \text{ for all } i, o, d. \end{array} \right. \end{array} \right\} \tag{11}$$

This yields a similar, but more tractable procedure.

Proposition 3. *Let $Q_d^* \ge 0$ be such that the optimal solution $q^* = (q_i^*)_{i\in I}$ to (11) yields multipliers $\gamma_d = 0$ associated with $\sum_i q_{id}^* = Q_d^*$ for all $d \in D$. Then the profile $q^* = (q_i^*)_{i\in I}$ is a Cournot-Nash equilibrium with partial coalitional strategies. Conversely, let $\bar{q} = (\bar{q}_i)_{i\in I} \ge 0$ be an equilibrium solution. Then the profile $\bar{q} = (\bar{q}_i)_{i\in I}$ solves (11) where $\sum_i \bar{q}_{id} = \bar{Q}_d$ for each d, and, moreover, in case $\bar{Q}_d > 0$, then $\gamma_d = 0$ for all d.*

Proof. For given numbers q_{Io} and q_{Id}, reduce problem (11) by first solving the linear problem in x :

$$\begin{array}{ll} \text{maximize} & -cx \\ \text{subject to} & \left\{ \begin{array}{l} \sum_d x_{od} \le q_{Io} \text{ for all } o, \\ \sum_o x_{od} \ge q_{Id} \ \text{ for all } d, \\ \text{and } \ x_{od} \ge 0 \text{ for all } o, d. \end{array} \right. \end{array}$$

This problem is equivalent to (3), for which $c(q_I)$ is the optimal value function. Inserting $c(q_I)$ for cx in (11), the problem becomes one of maximizing with respect to q only. Now recall that $c(q_I) = u(q)$ and $\partial u(q)/\partial q_i = \Lambda(q) \ni \lambda$ (Definition 1 and Proposition 1). Thus the Kuhn-Tucker conditions corresponding to the reduced problem are the same as for program (9). Then invoke Proposition 2. □

In contrast to the previous procedure, one need not compute the optimal λ corresponding to each adjusted value Q_d when searching for those values that yield multipliers γ_d equal to zero. The appropriate λ at each adjustment stage will here be solved for implicitly.

3.2. A numerical instance. Consider an industry of three firms which supply customers at two destinations from one production site each, i.e., $i = o$, $q_i^{od} = q_{id}$, and $q_i = (q_{id})_{d\in D}$. The demand functions $p_d(Q_d)$, $d \in D = \{1, 2\}$, and production cost functions $f_i(q_i)$, $i \in I = \{1, 2, 3\}$, for this example are

$$p_1 = 5000^{\frac{1}{1.1}} Q_1^{-\frac{1}{1.1}},\ p_2 = 4000^{\frac{1}{1.1}} Q_2^{-\frac{1}{1.1}} \text{ and}$$

$$f_1 = 10q_1 + 0.14q_1^{1.8},\ f_2 = 8q_2 + 0.12q_2^{1.9},\ f_3 = 6q_3 + 0.1q_3^2.$$

The matrix of unit transportation cost coefficients c_{od} is

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}$$

The equilibrium individual supply levels $q_i = (q_{id})_{d\in D}$, satisfying either (9) and (11), are then $\bar{q}_1 = (44.909, 38.145)$, $\bar{q}_2 = (37.552, 31.619)$, and $\bar{q}_3 = (36.710, 30.872)$, which yield equilibrium industry output $\bar{Q}_1 = 119.171$ and $\bar{Q}_2 = 100.636$ with associated multipliers $\gamma_1 = 0.0004$ and $\gamma_2 = 0.00005$, respectively.

The cost minimizing distribution pattern

firm i:	1	2	3
$d = 1$	83.05	36.12	0
$d = 2$	0	33.05	67.58

of the equilibrium production levels $\sum_d \bar{q}_{1d} = 83.05$, $\sum_d \bar{q}_{2d} = 69.17$, and $\sum_d \bar{q}_{3d} = 67.58$, yields joint transportation cost $c(\bar{q}_I) = 325.1$ and optimal dual variables $\lambda_o = (2, 0, 1)$ and $\lambda_d = (3, 2)$. Splitting $c(\bar{q}_I)$ in accordance with the core allocation defined in (6), the resulting individual shares of joint transportation costs and individual (overall) profits are

firm i:	1	2	3
u_i	44.92	175.89	104.29
$\bar{\Pi}_i$	1151.9	915.9	1008.1

Exchanging the concerted transportation problem in (11) for the individual transportation problems in autarky, we can compute the standard spatial Cournot-Nash equilibrium for the specifications given above. Optimal supply levels $\hat{q}_i = (\hat{q}_{id})_{d\in D}$ are then $\hat{q}_1 = (47.535, 30.016)$, $\hat{q}_2 = (37.851, 32.532)$, and $\hat{q}_3 = (31.231, 34.069)$, which yield equilibrium industry output $\hat{Q}_1 = 116.616$ and $\hat{Q}_2 = 96.616$. Total output thus decreases if firms ship their own supply. The corresponding individual transportation costs $c\hat{x}_i$ and overall individual profits in this case are

firm i:	1	2	3
$c\hat{x}_i$	137.58	178.62	158.99
$\hat{\Pi}_i$	1007.8	925.7	924.6

As we can see, firms 1 and 3 increase their profits under cooperative distribution, whereas firm 2 is better off in the non-cooperative case.

4. Conclusion

The object of this paper was to analyze a spatial Cournot industry where the subsequent distribution of goods to consumers was arranged via the grand coalition, implying a two-stage game with both non-cooperative and cooperative strategies. An equilibrium for the overall game was established and characterized, and methods for determining an equilibrium solution provided. Furthermore, a numerical illustration highlights an interesting consequence of joint distribution in this context: Although transportation activities were competitive already at the outset (in autarky), i.e., by means of being exogenously priced, agreeing to distribute goods jointly increases total industry output − and, consequently, lowers the market price for the good. For authorities eager to increase welfare by introducing competition in network economies, it may be worth noting that appropriately designed regulation schemes may improve welfare beyond the scope of the invisible hand.

An alternative approach could be to analyze the overall game as a cooperative game, as did Sherali and Rajan (1986) for the oligopolistic (one-stage), non-spatial case. Then core solutions are not guaranteed. Moreover, the game would neither be one of orthogonal coalitions. Consequently, one should seek solutions in the α or β-core (see Shubik 1982). Anyway, when the parameters of demand and cost functions are such that overall individual profit increases, one would expect that firms have incentives to establish the grand coalition. Under which conditions this will happen is left for a subsequent paper.

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